Large Scale Geometry of Graphs of Polynomial Growth

Joint work with Anton Bernshteyn

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McGill Descriptive Dynamics and Combinatorics Seminar
Dec 5, 2023

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- Polynomial growth $\iff \rho_{\mathsf{ex}}(G) < \infty \iff \rho_{\mathsf{as}}(G) < \infty$.

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- Polynomial growth $\iff \rho_{\mathsf{ex}}(G) < \infty \iff \rho_{\mathsf{as}}(G) < \infty$.
- $\rho_{\mathsf{ex}}(G) \geqslant \rho_{\mathsf{as}}(G)$ for all G.
- If G is finite, then $\rho_{\mathsf{as}}(G) = 0$, while $\rho_{\mathsf{ex}}(G)$ can be arbitrarily large.

Examples: Grid_n and $\mathsf{Grid}_{n,\infty}$

G	V(G)	E(G)
Grid_n	$\mathbb{Z}^n$	$\{uv : u, v \in \mathbb{Z}^n, \|u - v\|_1 = 1\}$
$\mathsf{Grid}_{n,\infty}$	$\mathbb{Z}^n$	$\{uv : u, v \in \mathbb{Z}^n, \|u - v\|_\infty = 1\}$

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$$\rho_{\mathsf{ex}} = \Theta(n), \quad \rho_{\mathsf{as}} = n.$$

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Theorem (Gromov '81)

A finitely generated group Γ is of polynomial growth if and only if it is virtually nilpotent.

Asymptotic dimension

asdim in terms of padded decompositions

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An r-padded decomposition of a locally finite graph G with m layers is a family $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ of partitions of $V(G)$ into finite sets of uniformly bounded diameter such that for all $u \in V(G)$, there is some $\mathcal{P}_i$ such that $B_G(u, r) \subseteq [u]_{\mathcal{P}_i}$.

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Definition (Gromov '93)

The asymptotic dimension of a locally finite graph G, in symbols $\mathrm{asdim}(G)$, is the minimum $d \in \mathbb{N}$ (if it exists) such that for every $r \in \mathbb{N}$, G has an r-padded decomposition with $d+1$ layers.

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$\mathrm{asdim}(\mathbb{R}) = 1$

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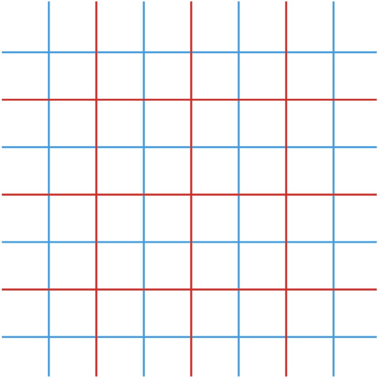

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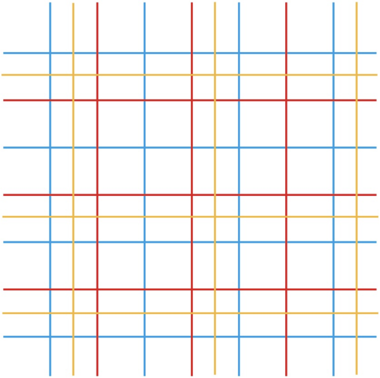

$\operatorname{asdim}^\alpha$ in terms of padded decompositions

asdim^α in terms of padded decompositions

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Definition

Let $\alpha > 1$. The asymptotic α-power dimension of a locally finite graph G, in symbols $\mathrm{asdim}^\alpha(G)$, is the minimum $d \in \mathbb{N}$ (if it exists) such that for every large $r \in \mathbb{N}$, G has an (r, r^α)-padded decomposition with $d + 1$ layers.

$$\mathrm{asdim}(G) \leq \mathrm{asdim}^\alpha(G).$$

asdim^{α} of graphs of polynomial growth

Theorem (Papasoglu '21)

Every graph G satisfies $\mathrm{asdim}(G) \leqslant \lfloor \rho_{\mathsf{as}}(G) \rfloor$.

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- We show some stronger result:

Theorem (Bernshteyn–Y.)

Every graph G satisfies $\operatorname{asdim}^\alpha(G) \leqslant \lfloor \rho_{\mathsf{as}}(G) \rfloor$ *for all* $\alpha > \dfrac{\lfloor \rho_{\mathsf{as}}(G) \rfloor + 1}{\lfloor \rho_{\mathsf{as}}(G) \rfloor + 1 - \rho_{\mathsf{as}}(G)}$.

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- Moreover, our proof approach also works in the setting of Borel graphs and yields a Borel version of this theorem.

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Example

Let Γ be a group with a finite generating set $S \subseteq \Gamma$.

For a Borel action $\Gamma \curvearrowright X$ on a Polish space X, define the Schreier graph Sch(X, S): $V = X$, $E = \{\{x, \sigma \cdot x\} : x \in X,\ \sigma \in S,\ \sigma \cdot x \neq x\}$.

Components of Sch(X, S) $\rightsquigarrow$ orbits of the action $\Gamma \curvearrowright X$.

If the action $\Gamma \curvearrowright X$ is free, every component of Sch(X, S) is isomorphic to the Cayley graph of Γ.

Borel asymptotic dimension

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The Borel asymptotic dimension of a locally finite Borel graph G, in symbols $\mathrm{asdim}_{\mathsf{B}}(G)$ is the minimum $d \in \mathbb{N}$ (if it exists) such that for every $r \in \mathbb{N}$, G has a Borel r-padded decomposition with $d + 1$ layers.

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- $\mathrm{asdim}(G) \leqslant \mathrm{asdim}_{\mathsf{B}}(G)$.
- If $\mathrm{asdim}_{\mathsf{B}}(G) < \infty$, then $\mathrm{asdim}_{\mathsf{B}}(G) = \mathrm{asdim}(G)$.

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Theorem (Bernshteyn–Y. '23)

Every locally finite Borel graph G satisfies $\mathrm{asdim}_{\mathsf{B}}^{\alpha}(G) \leqslant \lfloor \rho_{\mathsf{as}}(G) \rfloor$ for all

$$\alpha > \frac{\lfloor \rho_{\mathsf{as}}(G) \rfloor + 1}{\lfloor \rho_{\mathsf{as}}(G) \rfloor + 1 - \rho_{\mathsf{as}}(G)}.$$

Hyperfiniteness and Marks' question

Definition (Weiss '84, Slaman–Steel '88)

A Borel graph is hyperfinite if there is an increasing sequence $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ of Borel subgraphs of G with finite components such that $G = \bigcup_{i=0}^{\infty} G_i$.

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Is every Borel graph of polynomial growth hyperfinite?

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Let G be a locally finite Borel graph. If $\mathrm{asdim}_{\mathsf{B}}(G) < \infty$, then G is hyperfinite.

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Since $\operatorname{asdim}_{\mathsf{B}}(G) \leq \operatorname{asdim}_{\mathsf{B}}^{\alpha}(G)$, we have

Corollary (Bernshteyn–Y. '23)

Every Borel graph of polynomial growth is hyperfinite.

Embeddings into grids

Embedding graphs of polynomial growth into grids

Conjecture (Levin–Linial–London–Rabinovich '95)

If G is a connected graph with $\rho_{\mathsf{ex}}(G) = \rho < \infty$, then

1. G is isomorphic to a subgraph of $\mathsf{Grid}_{n,\infty}$ for some $n < \infty$;
2. moreover, one can take $n = O(\rho)$.

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Question (Papasoglu '21)

Let G be a graph of polynomial growth rate ρ. Is there a coarse embedding $f : G \to \mathsf{Grid}_{n,\infty}$ with $n = O(\rho \log \rho)$?

Coarse embeddings

Definition (Gromov '93)

Given a pair of metric spaces (X, d_X), (Y, d_Y), a mapping $f\colon X \to Y$ is called a coarse embedding if there exist non-decreasing functions b, $B\colon [0,\infty] \to [0,\infty]$ such that:

- $b(\infty) = \infty$ and $B(x) < \infty$ for $x < \infty$;
- for all u, $v \in X$, $b(d_X(u,v)) \leqslant d_Y(f(u), f(v)) \leqslant B(d_X(u,v))$.

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- Let G be the Baumslag-Solitar group $BS(1,2) = \langle a, t | t^{-1}at = a^2 \rangle$. Let H be $\langle a \rangle$. Then the inclusion $i\colon H \to G$ is a coarse embedding:
  We can take $b(x) = \log x$ and $B(x) = x$.

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  We can take $b(x) = \log x$ and $B(x) = x$.
- Generally, let G be a finitely generated group and let H be a finitely generated subgroup of G. Then the inclusion $i\colon H \to G$ is a coarse embedding.
- A coarse embedding may be not injective, but it is asymptotically injective: preimages of points have uniformly bounded diameter.

Coarse embeddings into $\mathsf{Grid}_{n,\infty}$

Coarse embedding: $b(d_X(u,v)) \leqslant d_Y(f(u), f(v)) \leqslant B(d_X(u,v))$.

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- $B(r) = r$ means that f is a contraction.
  $\rightsquigarrow$ if $u \sim v$ in G, then $f(u) = f(v)$ or $f(u) \sim f(v)$ in $\mathsf{Grid}_{n,\infty}$

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The last result strengthens the Krauthgamer–Lee theorem.

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Let $\mathbb{Z}^n \curvearrowright 2^{\mathbb{Z}^n}$ be the Bernoulli shift action of $\mathbb{Z}^n$.

Let $X_n \subseteq 2^{\mathbb{Z}^n}$ be the free part of this action.

Define $\mathsf{ShiftGrid}_{n,\infty} := \mathsf{Sch}(X_n, \{\sigma \in \mathbb{Z}^n : \|\sigma\|_\infty = 1\})$.

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$\{\mathsf{ShiftGrid}_{n,\infty}\}_{n=1}^{\infty}$ are universal for Borel graphs of polynomial growth!

Application to hyperfiniteness

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PROOF SKETCH. Let G be a Borel graph of polynomial growth.

There is a Borel injection from G to $\mathsf{ShiftGrid}_{n,\infty}$ for some $n < \infty$.

By Jackson–Kechris–Louveau, $\mathsf{ShiftGrid}_{n,\infty}$ is hyperfinite.

Hyperfiniteness can be "pulled back" via an injection. $\square$

Application to the existence of toasts

Definition (r-toast)

Let G be a Borel graph. For $r \in \mathbb{N}$, a Borel family $\mathcal{T} \subseteq [V(G)]^{<\infty}$ of finite sets is an r-**toast** if the following two conditions hold:

1. for every edge $uv \in E(G)$, there is some $K \in \mathcal{T}$ such that u, $v \in K$, and
2. for distinct K, $L \in \mathcal{T}$, we have either $B_G(K, r) \cap L = \emptyset$, $B_G(K, r) \subseteq L$, or $B_G(L, r) \subseteq K$.

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By a result of Gao–Jackson–Krohne–Seward, there is an r-toast $\mathcal{T}^* \subseteq [\mathsf{Free}(2^{\mathbb{Z}^n})]^{<\infty}$ for $\mathsf{ShiftGrid}_{n,\infty}$.

It suffices to verify that $\mathcal{T} := \{K \cap V(G) \,:\, K \in \mathcal{T}^*\}$ is an r-toast for G. $\square$

Embedding graphs of polynomial growth into grids

Conjecture (Levin–Linial–London–Rabinovich '95)

If G is a connected graph with $\rho_{\mathsf{ex}}(G) = \rho < \infty$, then

1. G is isomorphic to a subgraph of $\mathsf{Grid}_{n,\infty}$ for some $n < \infty$;
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If G is a Borel graph with $\rho_{\mathsf{as}}(G) = \rho < \infty$ and $\operatorname{asdim}^{\alpha}_{\mathsf{B}}(G) = k$ with some $\alpha > 1$, then for every $0 < \epsilon < 1/\alpha$ there is a Borel coarse embedding f of G into $\mathsf{ShiftGrid}_{n,\infty}$ with $n = O_\epsilon(\alpha\rho)$, and

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When k is small for small α, we have nice result.

$\mathrm{asdim}^{\mathsf{N}}$ and $\mathrm{asdim}_{\mathsf{B}}^{\mathsf{N}}$

Definition (Assouad '82)

Let $\alpha > 1$. The Nagata dimension of a locally finite graph G, in symbols $\mathrm{asdim}^{\mathsf{N}}(G)$, is the minimum $d \in \mathbb{N}$ (if it exists) such that there exists $c > 0$ satisfying "for every large $r \in \mathbb{N}$, G has an (r, cr)-padded decomposition with $d + 1$ layers."

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Theorem (Papasoglu '21)

There exists some graph G with $\rho_{\mathsf{as}}(G) = 1$ and $\mathrm{asdim}^{\mathsf{N}}(G) = \infty$.

Minor-excluded Graphs

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Planar graphs, outerplanar graphs, trees, ..

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Theorem (Liu '23, Distel '23)

If G is a graph excluding a fixed finite minor, then $\mathrm{asdim}^{\mathsf{N}}(G) \leq 2$.

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Apply techniques in [Conley–Jackson–Marks–Seward–Tucker-Drob '20], we have

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If $\mathrm{asdim}_{\mathsf{B}}(G) < \infty$, then $\mathrm{asdim}^{\mathsf{N}}_{\mathsf{B}}(G) = \mathrm{asdim}^{\mathsf{N}}(G)$.

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Open problems

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- Hyperfiniteness of Borel graphs of subexponential growth?

THANK YOU
Q&A

Jing Yu
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